Notes on the Classification of Vector Bundles

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Rob,

The only way I can understand these things is by reconstructing the ideas. I am very happy because I had never worked out the bundle classification stuff out in detail. I learned a long time ago from Singer that it is not necessary to do things in the most generality to try to understand what is going on. For example, the exact homotopy sequence is a theorem of Topology but it is easier to understand if you are in Geometry where there is a connection as in the case $0 \to H \to G \to G/H \to 0$.

The notes are very rough. I have added more stuff to Section 1. I want to work out here how to compute $H^2(M, \mathbb{Z}_N)$ in the case of four manifolds so I worked out the associated problem in two dimensions by working out $H^1(M, \mathbb{Z}_N)$ for the torus and the Klein bottle. I still have to get to $H^2(M, \mathbb{Z}_N)$ for $\dim M = 4$. In Sections 2 and 3 I use as a warm up to go on to more advanced stuff and later on I will add some stuff here to see how much easier it is if you have a connection. Finally the stuff of interest is in Section 4. At the end I explain the origin of classifying spaces in Section 7.

I have taken to writing notes in LaTeX because I can never find old hand written notes.

Orlando

[Feb. 11] Corrected some typos and slightly changed some notation and added some miscellaneous improvements. I think I see the light at the end of the tunnel to make contact with what you sent me on $\mathbb{Z}_N$ monopoles. I have to finish something else this week so I will get back to this later.

[Feb. 12] Corrected more typos some critical.

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Contents

1 Basics

I am going to try to develop the classification of G-bundles over a four manifold. You don't need all the general machinery.

We are on a connected compact manifold M. We implicitly assume that we have chosen a "good cover" $\{V_\alpha\}$ on M.

We discuss two examples. In the orientation example (Section 2) we are starting with a larger group $\mathrm{O}(n)$ and reducing to a smaller group $\mathrm{SO}(n)$. In the spin structure example (Section 3) we start with a smaller group $\mathrm{SO}(n)$ and try to lift things to a larger group $\mathrm{Spin}(n)$.

Let's make some basic homological and cohomological observations. For the moment $\dim M = n$ and we assume M is orientable. It is a fact that the homology groups are of the form $H_k(M,\mathbb{Z}) = \mathbf{F}_k \oplus \mathbf{T}_k$ where

$$\begin{aligned}
\mathbf{F}_k &= \mathbb{Z} \oplus \cdots \oplus \mathbb{Z}\,, \\
\mathbf{T}_k &= \mathbb{Z}_{n_1} \oplus \cdots \oplus \mathbb{Z}_{n_{r_k}}\,.
\end{aligned}$$

$\mathbf{F}_k$ is called the free part and $\mathbf{T}_k$ is called the torsion part. The universal coefficient theorem tells you that the cohomology is given by

$$H^k(M,\mathbb{Z}) = \mathbf{F}_k \oplus \mathbf{T}_{k-1}\,. \tag{1.1}$$

Poincaré duality tells you that $H_k(M,\mathbb{Z}) \approx H^{n-k}(M,\mathbb{Z})$. Putting this together with the universal coefficients theorem we have that $\mathbf{F}_n \approx \mathbf{F}_{n-k}$ and $\mathbf{T}_k \approx \mathbf{T}_{n-k-1}$. For a connected compact manifold $H_0(M,\mathbb{Z}) = \mathbb{Z}$ and $H_n(M,\mathbb{Z}) = \mathbb{Z}$. Note that $\mathbf{T}_0 = 0$.

Let's specialize to the case of $\dim M = 4$ and orientable. In this case we have that $\mathbf{F}_k \approx \mathbf{F}_{4-k}$ and $\mathbf{T}_k \approx \mathbf{T}_{3-k}$. We can make a table:

$$\begin{aligned}
H^0(M,\mathbb{Z}) &= \mathbb{Z}\,, \\
H^1(M,\mathbb{Z}) &= \mathbf{F}_1\,, \\
H^2(M,\mathbb{Z}) &= \mathbf{F}_2 \oplus \mathbf{T}_1\,, \\
H^3(M,\mathbb{Z}) &= \mathbf{F}_1 \oplus \mathbf{T}_1\,, \\
H^4(M,\mathbb{Z}) &= \mathbb{Z}\,.
\end{aligned} \tag{1.2}$$

Let's specialize to the case of $\dim \Sigma = 3$ and orientable. In this case we have that $\mathbf{F}_k(\Sigma) \approx \mathbf{F}_{3-k}(\Sigma)$ and $\mathbf{T}_k(\Sigma) \approx \mathbf{T}_{2-k}(\Sigma)$. We can make a table:

$$\begin{aligned}
H^0(\Sigma,\mathbb{Z}) &= \mathbb{Z}\,, \\
H^1(\Sigma,\mathbb{Z}) &= \mathbf{F}_1(\Sigma)\,, \\
H^2(\Sigma,\mathbb{Z}) &= \mathbf{F}_1(\Sigma) \oplus \mathbf{T}_1(\Sigma)\,, \\
H^3(\Sigma,\mathbb{Z}) &= \mathbb{Z}\,.
\end{aligned}$$

For a 4-manifold of the form $M = S^1 \times \Sigma$ we have

$$\begin{aligned}
H^0(M, \mathbb{Z}) &= \mathbb{Z}\,, \\
H^1(M, \mathbb{Z}) &= H^1(\Sigma, \mathbb{Z}) \oplus \mathbb{Z}\,, \\
&= \mathbf{F}_1(\Sigma) \oplus \mathbb{Z}\,, \\
H^2(M, \mathbb{Z}) &= H^2(\Sigma, \mathbb{Z}) \oplus H^1(\Sigma, \mathbb{Z})\,, \\
&= \mathbf{F}_1(\Sigma) \oplus \mathbf{F}_1(\Sigma) \oplus \mathbf{T}_1(\Sigma)\,, \\
H^3(M, \mathbb{Z}) &= \mathbb{Z} \oplus H^2(\Sigma, \mathbb{Z})\,, \\
&= \mathbb{Z} \oplus \mathbf{F}_1(\Sigma) \oplus \mathbf{T}_1(\Sigma)\,, \\
H^4(M, \mathbb{Z}) &= \mathbb{Z}\,.
\end{aligned} \tag{1.3}$$

This table agrees with equation (1.2). Note that we learned that $\mathbf{F}_1(M) \approx \mathbf{F}_1(\Sigma) \oplus \mathbb{Z}$ and $\mathbf{F}_2(M) \approx \mathbf{F}_1(\Sigma) \oplus \mathbf{F}_1(\Sigma)$ and $\mathbf{T}_1(M) \approx \mathbf{T}_1(\Sigma)$.

For the type of manifold we consider $\Sigma = S^3, S^1 \times S^2, S^1 \times S^1 \times S^1$ there is no torsion so things simplify. For example if $\Sigma = S^3$ then

$$\begin{aligned}
H^0(S^1 \times S^3, \mathbb{Z}) &= \mathbb{Z}\,, \\
H^1(S^1 \times S^3, \mathbb{Z}) &= \mathbb{Z}\,, \\
H^2(S^1 \times S^3, \mathbb{Z}) &= 0\,, \\
H^3(S^1 \times S^3, \mathbb{Z}) &= \mathbb{Z}\,, \\
H^4(S^1 \times S^3, \mathbb{Z}) &= \mathbb{Z}\,.
\end{aligned} \tag{1.4}$$

For the type of manifold 'tHooft considered $\Sigma = (S^1)^3$ then

$$\begin{aligned}
H^0(S^1 \times (S^1)^3, \mathbb{Z}) &= \mathbb{Z}\,, \\
H^1(S^1 \times (S^1)^3, \mathbb{Z}) &= \mathbb{Z}^4\,, \\
H^2(S^1 \times (S^1)^3, \mathbb{Z}) &= \mathbb{Z}^6\,, \\
H^3(S^1 \times (S^1)^3, \mathbb{Z}) &= \mathbb{Z}^4\,, \\
H^4(S^1 \times (S^1)^3, \mathbb{Z}) &= \mathbb{Z}\,.
\end{aligned} \tag{1.5}$$

Finally to compute $H^k(M, \mathbb{Z}_N)$ we need the long exact cohomology sequence. Given the short exact sequence

$$0 \to \mathbb{Z} \xrightarrow{\quad N \quad} \mathbb{Z} \xrightarrow{\ \bmod N\ } \mathbb{Z}_N \to 0$$

there is a long exact sequence in cohomology.

$$\cdots \to H^p(M,\mathbb{Z}) \xrightarrow{\bmod N} H^p(M,\mathbb{Z}_N) \xrightarrow{\vartheta} H^{p+1}(M,\mathbb{Z}) \xrightarrow{N} H^p(M,\mathbb{Z}) \to \cdots .$$

The piece that interests us is

$$\cdots \to H^2(M,\mathbb{Z}) \xrightarrow{\bmod N} H^2(M,\mathbb{Z}_N) \xrightarrow{\vartheta} H^3(M,\mathbb{Z}) \xrightarrow{N} H^3(M,\mathbb{Z}) \to \cdots , \tag{1.6}$$

where ϑ is the Bockstein homomorphism. What is the Bockstein homomorphism? I am going to drop the "M" to simplify notation. The p-cochains in M with integer coefficients will be denoted by $C^p(\mathbb{Z})$, etc. The short exact sequence induces a short exact sequence on cochains. The Cech differential δ commutes with the group actions. Thus we have

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C^{p+1}(\mathbb{Z}) & \xrightarrow{N} & C^{p+1}(\mathbb{Z}) & \xrightarrow{\bmod N} & C^{p+1}(\mathbb{Z}_N) & \longrightarrow & 0 \\ & & \delta\Big\uparrow & & \delta\Big\uparrow & & \delta\Big\uparrow & & \\ 0 & \longrightarrow & C^{p}(\mathbb{Z}) & \xrightarrow{N} & C^{p}(\mathbb{Z}) & \xrightarrow{\bmod N} & C^{p}(\mathbb{Z}_N) & \longrightarrow & 0 \end{array} \tag{1.7}$$

Let's construct the Bockstein homomorphism. Let $[\nu] \in H^p(\mathbb{Z}_N)$, I can represent this cohomology class[1] by a cochain in $\nu \in C^p(\mathbb{Z}_N)$ and think of it in two equivalent ways. I have a collection of integers modulo N with $(p+1)$ indices $\{n_{\alpha_0\alpha_1\ldots\alpha_p} \bmod N\}$ or I can write this as $\{\exp(2\pi i n_{\alpha_0\alpha_1\ldots\alpha_p}/N)\}$. The cocycle condition may be written as $\delta\{n_{\alpha_0\alpha_1\ldots\alpha_p} \bmod N\} = 0 \bmod N$. Note that $\{n_{\alpha_0\alpha_1\ldots\alpha_p}\} \in C^p(\mathbb{Z})$ and that $\delta\{n_{\alpha_0\alpha_1\ldots\alpha_p}\} = N\{m_{\alpha_0\alpha_1\ldots\alpha_p\alpha_{p+1}}\}$ where $\{m_{\alpha_0\alpha_1\ldots\alpha_p\alpha_{p+1}}\} \in C^p(\mathbb{Z})$. It is straightforward to verify that $\delta\{m_{\alpha_0\alpha_1\ldots\alpha_p\alpha_{p+1}}\} = 0$. This means that to the cohomology class $[\nu] \in H^p(\mathbb{Z}_N)$ I can assign a cohomology class $[m_{\alpha_0\alpha_1\ldots\alpha_p\alpha_{p+1}}] \in H^{p+1}(\mathbb{Z})$. You can show that all this is does not depend on the representatives chosen for the cohomology classes. The upshot of this is that we can schematically write for the Bockstein map:

$$\{m\} = \frac{1}{N}\,\delta\{n\}\,. \tag{1.8}$$

To make contact with something we know we look at the cohomology classes needed to define a line bundle. A line bundle is defined by a 1-cocycle $\{\varphi_{\alpha\beta}\}$ that satisfies $\varphi_{\alpha\beta}\varphi_{\beta\gamma}\varphi_{\gamma\alpha} = 1$. Let $\mathcal{F}^*$ be the "sheaf of non-vanishing complex valued functions", i.e., local functions that don't vanish. Let $\mathcal{F}$ be the sheaf of complex valued functions. There is a short exact sequence $0 \to 2\pi i\mathbb{Z} \to \mathcal{F} \xrightarrow{\exp} \mathcal{F}^* \to 0$. From my Cech cohomology paper you know that $H^p(\mathcal{F}) = 0$ for $p > 0$. Given the 1-cocycle $\{\varphi_{\alpha\beta}\}$, the short exact sequence tells us we can construct a 1-cochain $\{\psi_{\alpha\beta}\} \in C^1(\mathcal{F})$ such that $e^{i\psi_{\alpha\beta}} = \varphi_{\alpha\beta}$. The cocycle condition on φ becomes $\exp(\theta_{\alpha\beta} + \theta_{\beta\gamma} + \theta_{\gamma\alpha}) = 1$. This

[1] I use the standard notation that equivalence class of ν is written ad $[\nu]$.

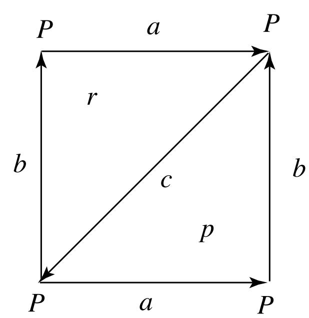

Figure 1: A torus represented as a square with opposite sides identified. There is one vertex P. There are two 2-simplices p and r, and three 1-simplices a, b and c. Note that $\partial P = 0$, $\partial a = \partial b = \partial c = 0$, and $\partial p = a+b+c$ and $\partial r = -(a+b+c)$. The fundamental 2-cycle of the torus is $p+r$ with $\partial(p+r) = 0$.

tells us that $\theta_{\alpha\beta} + \theta_{\beta\gamma} + \theta_{\gamma\alpha} = 2\pi i n_{\alpha\beta\gamma}$ where $\{n_{\alpha\beta\gamma}\} \in C^2(\mathbb{Z})$. You can verify that this integral 2-cochain is closed. The associated long exact sequence is

$$\begin{array}{ccccccc}
H^1(\mathcal{F}) & \xrightarrow{\exp} & H^1(\mathcal{F}^*) & \xrightarrow{\vartheta} & H^2(\mathbb{Z}) & \longrightarrow & H^2(\mathcal{F}) \\
\| & & \| & & \| & & \| \\
0 & \xrightarrow{\exp} & H^1(\mathcal{F}^*) & \xrightarrow{\vartheta} & H^2(\mathbb{Z}) & \longrightarrow & 0
\end{array}$$

The bottom row tells us that $H^1(M, \mathcal{F}^*) \approx H^2(M, \mathbb{Z})$ and so line bundles are topologically classified by their first Chern class. The lesson here is that even though $H^1(\mathcal{F}) = 0$ we can have $H^1(\mathcal{F}^*) \neq 0$.

The reason for the last comment is that in the piece of the long exact cohomology sequence that is of interest to us (1.6) it may be possible for $H^2(M, \mathbb{Z}) = 0$ but $H^2(M, \mathbb{Z}_N) \neq 0$ just the stuff involving $\mathcal{F}$ and $\mathcal{F}^*$. Also to connect to the above I remind you that $(n \mod N)$ is the same as $e^{2\pi i n/N}$.

1.1 Homology and Cohomology of a Torus

As a warm-up exercise let's compute the homology of a torus with simplicial decomposition given in Figure 1. In a sense that I will make clear in the example that follows it is easier for $H^2(M, \mathbb{Z}_N)$ to be non-zero compared to $H^2(M, \mathbb{Z})$ analogous to the line bundle case for $H^1(M, \mathcal{F}^*)$ and $H^1(M, \mathcal{F})$. I want to work out an example using the torus $M = \mathbb{T}^2$ to illustrate my point.

First let's compute the homology of a torus. Let the boundary operator acting on k-chains be denoted by $\partial_k : C_k(M,\mathbb{Z}) \to C_{k-1}(M,\mathbb{Z})$. The homology groups are given by $H_k(M,\mathbb{Z}) = (\ker \partial_k)/(\operatorname{im} \partial_{k+1})$.

1. [Computation of $H_0(M,\mathbb{Z})$] Clearly $\ker \partial_0 = \{kP \mid k \in \mathbb{Z}\}$. Since $\partial a = \partial b = \partial c = 0$ we have that $\operatorname{im} \partial_1 = 0$. From this we see that $H_0(M,\mathbb{Z}) = \mathbb{Z}$.

2. [Computation of $H_1(M,\mathbb{Z})$] We see that $\ker \partial_1 = \{k_a a + k_b b + k_c c \mid k_a, k_b, k_c \in \mathbb{Z}\} = \mathbb{Z}^3$. Next we see that $\operatorname{im} \partial_2 = \{\partial(k_p p + k_r r) = (k_p - k_r)(a+b+c) \mid k_p, k_r \in \mathbb{Z}\} = \mathbb{Z}$. Note that we can rewrite $k_a a + k_b b + k_c c = (k_a - k_c)a + (k_b - k_c)b + k_c(a+b+c)$. The last term is $\operatorname{im} \partial_2$, once k_c is given then $k_a - k_c$ and $k_b - k_c$ can be freely specified so we see that $H_1(M,\mathbb{Z}) = \mathbb{Z}^2$ and it is generated by the $[a]$ and $[b]$ cycles.

3. [Computation of $H_2(M,\mathbb{Z})$] We know that there are no 3-chains so $H_2(M,\mathbb{Z}) = \ker \partial_2$. Since $\partial(k_p p + k_r r) = (k_p - k_r)(a+b+c)$ we see that $H_2(M,\mathbb{Z}) = \ker \partial_2 = \{k(p+r) \mid k \in \mathbb{Z}\} = \mathbb{Z}$ and $H_2(M,\mathbb{Z})$ generated by the fundamental cycle $[p+r]$.

Now we move to computing the cohomology. Remember that the cochains are the linear functionals: $C^p(M,\mathbb{Z}) = \{\alpha : C_p(M,\mathbb{Z}) \to \mathbb{Z}\}$. For 0-chain P we have a linear functional Π. For the 1-chains we have the dual basis α, β and γ. For the 2-cochains we have dual basis π and ρ. If we denote the differential by $\delta_k : C^k(M,\mathbb{Z}) \to C^{k+1}(M,\mathbb{Z})$ then $H^k(M,\mathbb{Z}) = (\ker \delta_k)/(\operatorname{im} \delta_{k-1})$. If $\lambda \in C^k(M,\mathbb{Z})$ then the differential δ_k is defined by $(\delta_k \lambda) = \lambda \circ \partial_{k+1}$.

1. [Computation of $H^0(M,\mathbb{Z})$] In this case we have that $H^0(M,\mathbb{Z}) = \ker \delta_0$. Now $(\delta\Pi)(k_a a + k_b b + k_c c) = \Pi(\partial(k_a a + k_b b + k_c c)) = 0$ which tells us that $H^0(M,\mathbb{Z}) = \ker \delta_0 = \{n\Pi \mid n \in \mathbb{Z}\} = \mathbb{Z}$.

2. [Computation of $H^1(M,\mathbb{Z})$] $\lambda \in C^1(M,\mathbb{Z})$ is of the form $\lambda = n_\alpha \alpha + n_\beta \beta + n_\gamma \gamma$. We want $\delta\lambda = 0$ which means that $\lambda \circ \partial_2 = 0$, i.e., $\lambda(\partial_2 p) = (n_\alpha + n_\beta + n_\gamma) = 0$ and $\lambda(\partial_2 r) = -(n_\alpha + n_\beta + n_\gamma) = 0$. Note that in computing $H^0(M,\mathbb{Z})$ we actually learned that $\operatorname{im} \delta_0 = 0$. Note that we can write $\lambda = n_\alpha(\alpha - \gamma) + n_\beta(\beta - \gamma) + (n_\alpha + n_\beta + n_\gamma)\gamma$ so the cocycles are $\ker \delta_1 = \{n_\alpha(\alpha - \gamma) + n_\beta(\beta - \gamma)\} = \mathbb{Z}^2$. Thus we have shown that $H^1(M,\mathbb{Z}) = \mathbb{Z}^2$.

3. [Computation of $H^2(M,\mathbb{Z})$] Since we are at the top we have that $\ker \delta_2 = C^2(M,\mathbb{Z})$. From the calculation of $H^1(M,\mathbb{Z})$ you see that $\operatorname{im} \delta_1 = \{(n_\alpha + n_\beta + n_\gamma)(\pi - \rho)\}$. Since we can write $k_\pi \pi + k_\rho \rho = k_\pi(\pi - \rho) + (k_\pi + k_\rho)\rho$ we see that $H^2(M,\mathbb{Z}) = \mathbb{Z}$.

The computation above could also be done using the following table

$$
\begin{gathered}
\delta\Pi = 0\,, \\
\delta\alpha = \pi - \rho\,, \quad \delta\beta = \pi - \rho\,, \quad \delta\gamma = \pi - \rho\,, \\
\delta\pi = 0\,, \qquad\qquad\qquad\qquad \delta\rho = 0\,.
\end{gathered}
\tag{1.9}
$$

which will be useful in what follows.

Next we compute cohomology with $\mathbb{Z}_N$ coefficients. A 1-cochain with values in an abelian group A is a linear transformation that assign to each integer 1-chain an element of A. For example $\lambda \in C^1(M, \mathbb{Z}_N)$ is given by $\lambda = u_\alpha \alpha + u_\beta \beta + u_\gamma \gamma$ where u_α, u_β and u_γ are in $\mathbb{Z}_N$. It is convenient to write $u_\alpha = n_\alpha \mod N$, etc. Let's compute $H^1(M, \mathbb{Z}_N)$. We have that $\delta\lambda = (n_\alpha + n_\beta + n_\gamma)(\pi - \rho) \mod N$ and λ may be written as $\lambda = n_\alpha(\alpha - \gamma) + n_\beta(\beta - \gamma) + (n_\alpha + n_\beta + n_\gamma)\gamma \mod N$. From this we see that $\ker \delta_1 = \{n_\alpha(\alpha - \gamma) + n_\beta(\beta - \gamma) \mod N\}$. Since $\operatorname{im} \delta_0 = 0$ we have that $H^2(M, \mathbb{Z}_N) = (\ker \delta_1)/(\operatorname{im} \delta_0) = \mathbb{Z}_N \oplus \mathbb{Z}_N$ and is generated by $[\alpha - \gamma]$ and $[\beta - \gamma]$.

Next we construct the Bockstein homomorphism $\vartheta : H^1(M, \mathbb{Z}_N) \to H^2(M, \mathbb{Z})$. Let $[\lambda] \in H^1(M, \mathbb{Z}_N)$. Choose a representative $\lambda = n_\alpha(\alpha - \gamma) + n_\beta(\beta - \gamma) \mod N \in C^1(M, \mathbb{Z}_N)$. This representative is the image of a chain $\mu = n_\alpha(\alpha - \gamma) + n_\beta(\beta - \gamma) \in C^1(M, \mathbb{Z})$, see the red $\mod N$ map in (1.7). Next we apply the blue δ operation in (1.7) and obtain $\delta\mu = 0$. This means that the inverse image of the green N map in (1.7) is zero and thus we conclude that $\vartheta[\lambda] = 0$. The relevant piece of the long exact sequence is

$$
\begin{array}{ccccccc}
H^1(M,\mathbb{Z}) & \xrightarrow{\mod N} & H^1(M,\mathbb{Z}_N) & \xrightarrow{\vartheta} & H^2(M,\mathbb{Z}) & \xrightarrow{N} & H^2(M,\mathbb{Z}) \\
\| & & \| & & \| & & \| \\
\mathbb{Z} \oplus \mathbb{Z} & \xrightarrow[\text{onto}]{\mod N} & \mathbb{Z}_N \oplus \mathbb{Z}_N & \xrightarrow{\vartheta = 0} & \mathbb{Z} & \xrightarrow[\text{1-1}]{N} & \mathbb{Z}
\end{array}
$$

The "arrow subscripts" are the consequences of the Bockstein map being zero.

1.2 Homology and Cohomology of a Klein Bottle

As another warm-up exercise let's compute the homology of a Klein bottle with simplicial decomposition given in Figure 2. I want to work out an example using the Klein bottle M to illustrate some subtleties.

First let's compute the homology.

1. [Computation of $H_0(M, \mathbb{Z})$] Clearly $\ker \partial_0 = \{kP \mid k \in \mathbb{Z}\}$. Since $\partial a = \partial b = \partial c = 0$ we have that $\operatorname{im} \partial_1 = 0$. From this we see that $H_0(M, \mathbb{Z}) = \mathbb{Z}$.

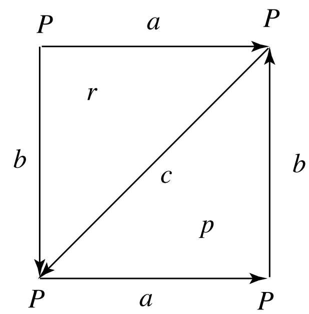

Figure 2: A Klein bottle represented as a square with opposite sides identified. There is one vertex P. There are two 2-simplices p and r, and three 1-simplices a, b and c. Note that $\partial P = 0$, $\partial a = \partial b = \partial c = 0$, and $\partial p = a + b + c$ and $\partial r = -a + b - c$.

2. [Computation of $H_1(M, \mathbb{Z})$] We see that $\ker \partial_1 = \{k_a a + k_b b + k_c c \mid k_a, k_b, k_c \in \mathbb{Z}\} = \mathbb{Z}^3$. Next we see that $\operatorname{im} \partial_2 = \{\partial(k_p p + k_r r) = k_p(a+b+c) + k_q(-a+b-c)\}$. We remark that $k_p(a + b + c) + k_q(-a + b - c) = (k_p - k_q)(a + b + c) + 2k_q b$, and especially that $k_p - k_q$ can be an arbitrary integer and that $2k_q$ can be an arbitrary even integer. We rewrite $k_a a + k_b b + k_c c = (k_a - k_c)a + (k_b - k_c)b + k_c(a + b + c)$. We see that $H_1(M, \mathbb{Z}) = (\ker \partial_1)/(\operatorname{im} \partial_2) = \mathbb{Z} \oplus \mathbb{Z}_2$ where $\mathbb{Z}$ is generated by $[a]$ and $\mathbb{Z}_2$ is generated by $[b]$. Here we have a torsion subgroup $\mathbb{Z}_2$ in the homology.
3. [Computation of $H_2(M, \mathbb{Z})$] We know that there are no 3-chains so $H_2(M, \mathbb{Z}) = \ker \partial_2$. Since $\partial(k_p p + k_r r) = k_p(a + b + c) + k_r(-a + b - c) \neq 0$ we see that $H_2(M, \mathbb{Z}) = \ker \partial_2 = 0$.

Now we move to computing the cohomology.

1. [Computation of $H^0(M, \mathbb{Z})$] In this case we have that $H^0(M, \mathbb{Z}) = \ker \delta_0$. Now $(\delta\Pi)(k_a a + k_b b + k_c c) = \Pi(\partial(k_a a + k_b b + k_c c)) = 0$ which tells us that $H^0(M, \mathbb{Z}) = \ker \delta_0 = \{n\Pi \mid n \in \mathbb{Z}\} = \mathbb{Z}$.
2. [Computation of $H^1(M, \mathbb{Z})$] $\lambda \in C^1(M, \mathbb{Z})$ is of the form $\lambda = n_\alpha \alpha + n_\beta \beta + n_\gamma \gamma$. We want $\delta\lambda = 0$ which means that $\lambda \circ \partial_2 = 0$, i.e., $\lambda(\partial_2 p) = n_\alpha + n_\beta + n_\gamma = 0$ and $\lambda(\partial_2 r) = -n_\alpha + n_\beta - n_\gamma = 0$. These equations tell you that $n_\alpha + n_\gamma = 0$ and $n_\beta = 0$. Note that in computing $H^0(M, \mathbb{Z})$ we actually learned that $\operatorname{im} \delta_0 = 0$. Note that we can write $\lambda = n_\alpha(\alpha - \gamma) + n_\beta \beta + (n_\alpha + n_\gamma)\gamma$ so the cocycles are $\ker \delta_1 = \{n_\alpha(\alpha - \gamma)\} = \mathbb{Z}$. Thus we have shown that $H^1(M, \mathbb{Z}) = \mathbb{Z}$. This is in agreement with the universal coefficients theorem (1.1).

3. [Computation of $H^2(M,\mathbb{Z})$] Since we are at the top we have that $\ker\delta_2 = C^2(M,\mathbb{Z})$. From the calculation of $H^1(M,\mathbb{Z})$ you see that $\operatorname{im}\delta_1 = \{(n_\alpha + n_\beta + n_\gamma)\pi + (-n_\alpha + n_\beta - n_\gamma)\rho\}$. We can write $(n_\alpha + n_\beta + n_\gamma)\pi + (-n_\alpha + n_\beta - n_\gamma)\rho = (n_\alpha + n_\beta + n_\gamma)(\pi - \rho) + 2n_\beta\rho$ and $k_\pi\pi + k_\rho\rho = k_\pi(\pi - \rho) + (k_\pi + k_\rho)\rho$. We note that $n_\alpha + n_\beta + n_\gamma$ is an arbitrary integer and $2n_\beta$ is an arbitrary even integer. From this we see that $H^2(M,\mathbb{Z}) = \mathbb{Z}_2$ where we can take the generator to be $[\rho]$. This result agrees with the universal coefficients theorem (1.1).

The computation above could also be done using the following table

$$
\begin{gathered}
\delta\Pi = 0\,, \\
\delta\alpha = \pi - \rho\,, \quad \delta\beta = \pi + \rho\,, \quad \delta\gamma = \pi - \rho\,, \\
\delta\pi = 0\,, \qquad\qquad\qquad\qquad \delta\rho = 0\,.
\end{gathered}
\tag{1.10}
$$

which will be useful in what follows.

Next we compute cohomology with $\mathbb{Z}_N$ coefficients. A For example, let $\lambda \in C^1(M,\mathbb{Z}_N)$ be given by $\lambda = n_\alpha\alpha + n_\beta\beta + n_\gamma\gamma \mod N$ We have that $\delta\lambda = (n_\alpha + n_\beta + n_\gamma)\pi + (-n_\alpha + n_\beta - n_\gamma)\rho \mod N$. From this we see that $\lambda \in \ker\delta_1$ if $n_\alpha + n_\beta + n_\gamma = 0 \mod N$ and $-n_\alpha + n_\beta - n_\gamma = 0 \mod N$. Next we note that $-n_\alpha + n_\beta - n_\gamma = 2n_\beta - (n_\alpha + n_\beta + n_\gamma)$ so the two condition for λ being in the kernel may be replaced by the two equivalent conditions $n_\alpha + n_\beta + n_\gamma = 0 \mod N$ and $2n_\beta = 0 \mod N$.

Since λ may be written as $\lambda = n_\alpha(\alpha - \gamma) + n_\beta(\beta - \gamma) + (n_\alpha + n_\beta + n_\gamma)\gamma \mod N$ we see that there are two cases to consider when determining $\ker\delta_1$. The simpler case is N odd and the more complicated case is N even.

First, we assume that N is odd and be observe that $\mathbb{Z}_N$ does not have a non-trivial element of order 2 and therefore we conclude that $n_\beta = 0 \mod N$. This means that $\lambda \in \ker\delta_1$ if $\lambda = n_\alpha(\alpha - \gamma) \mod N$. We see that $H^1(M,\mathbb{Z}_N) = \mathbb{Z}_N$ and is generated by $[\alpha - \gamma]$. It is also interesting to construct $H^2(M,\mathbb{Z}_N)$ because it fits nicely into the long exact cohomology sequence as we will see shortly. We note that $H^2(M,\mathbb{Z}_N) = C^2(M,\mathbb{Z}_N)/\delta_1 C^1(M,\mathbb{Z}_N)$. Note that $n_\pi\pi + n_\rho\rho \mod \mathbb{Z}_N = n_\pi(\pi - \rho) + (n_\rho + n_\pi)\rho \mod \mathbb{Z}_N$ and $\delta\lambda = (n_\alpha + n_\beta + n_\gamma)(\pi - \rho) + 2n_\beta\rho \mod N$. Since N is odd, elements of type $2n_\beta \mod N$ give you all of $\mathbb{Z}_N$ and we conclude that $H^2(M,\mathbb{Z}_N) = 0$. Next we construct the Bockstein homomorphism $\vartheta : H^1(M,\mathbb{Z}_N) \to H^2(M,\mathbb{Z})$. Let $[\lambda] \in H^1(M,\mathbb{Z}_N)$. Choose a representative $\lambda = n_\alpha(\alpha - \gamma) \mod N \in C^1(M,\mathbb{Z}_N)$. This representative is the image of a chain $\mu = n_\alpha(\alpha - \gamma) \in C^1(M,\mathbb{Z})$, see the red $\mod N$ map in (1.7). Next we apply the blue δ operation in (1.7) and obtain $\delta\mu = 0$. This means that the inverse image of the green N map in (1.7) is zero and thus we conclude that $\vartheta[\lambda] = 0$. The relevant piece of the long exact sequence is

$$
\begin{tikzcd}
H^1(M,\mathbb{Z}) \arrow[r, "\text{mod } N"] \arrow[d, equal] & H^1(M,\mathbb{Z}_N) \arrow[r, "\vartheta"] \arrow[d, equal] & H^2(M,\mathbb{Z}) \arrow[r, "N"] \arrow[d, equal] & H^2(M,\mathbb{Z}) \arrow[r, "\text{mod } N"] \arrow[d, equal] & H^2(M,\mathbb{Z}_N) \arrow[d, equal] \\
\mathbb{Z} \arrow[r, "\text{mod } N", "\text{onto}"'] & \mathbb{Z}_N \arrow[r, "\vartheta=0"] & \mathbb{Z}_2 \arrow[r, "N", "\text{1-1, onto}"'] & \mathbb{Z}_2 \arrow[r, "\text{mod } N", "\text{onto}"'] & 0
\end{tikzcd}
$$

From algebra we know that multiplication by odd N is an isomorphism in $\mathbb{Z}_2$ and this is in agreement with our figure. Of course we could have taken the long exact sequence and concluded that Bockstein was zero without the computation because the only homomorphism from $\mathbb{Z}_N$ to $\mathbb{Z}_2$ is the zero homomorphism if N is odd.

Secondly we consider N even which has and order 2 subgroup isomorphic to $\mathbb{Z}_2$ generated by $\{0 \mod N, \frac{1}{2}N \mod N\}$. We see that $H^1(M,\mathbb{Z}_N) = \mathbb{Z}_N \oplus \mathbb{Z}_N/\mathbb{Z}_2 \approx \mathbb{Z}_N \oplus \mathbb{Z}_{N/2}$ where $\mathbb{Z}_N$ is generated by $[\alpha - \gamma]$ and $\mathbb{Z}_{N/2}$ is generated by $[\beta - \gamma]$. Finally for completeness we calculate $H^2(M,\mathbb{Z}_N)$. In a previous calculation we used $n_\pi \pi + n_\rho \rho \mod \mathbb{Z}_N = n_\pi(\pi - \rho) + (n_\rho + n_\pi)\rho \mod \mathbb{Z}_N$ and $\delta\lambda = (n_\alpha + n_\beta + n_\gamma)(\pi - \rho) + 2n_\beta \rho \mod N$. The difference is that N is now even and elements of type $2n_\beta \mod N$ generate a $\mathbb{Z}_{N/2}$ subgroup. So we conclude that $H^2(M,\mathbb{Z}_N) = \mathbb{Z}_N/\mathbb{Z}_{N/2} \approx \mathbb{Z}_2$. I don't want to spend more time on this stuff but let me write down the relevant part of the long exact cohomology sequence.

$$
\begin{tikzcd}
H^1(M,\mathbb{Z}) \arrow[r, "\text{mod } N"] \arrow[d, equal] & H^1(M,\mathbb{Z}_N) \arrow[r, "\vartheta"] \arrow[d, equal] & H^2(M,\mathbb{Z}) \arrow[r, "N"] \arrow[d, equal] & H^2(M,\mathbb{Z}) \arrow[r, "\text{mod } N"] \arrow[d, equal] & H^2(M,\mathbb{Z}_N) \arrow[d, equal] \\
\mathbb{Z} \arrow[r, "\text{mod } N"] & \mathbb{Z}_N \oplus \mathbb{Z}_{N/2} \arrow[r, "\vartheta"] & \mathbb{Z}_2 \arrow[r, "N"] & \mathbb{Z}_2 \arrow[r, "\text{mod } N"] & \mathbb{Z}_2
\end{tikzcd}
$$

This table agrees with a theorem in algebraic topology [1, Proposition 3E.3 part c] that states that if p is a prime then a $\mathbb{Z}_p$ summand of $H^k(M,\mathbb{Z})$ gives $\mathbb{Z}_p$ summands of $H^{k-1}(M,\mathbb{Z}_p)$ and $H^k(M,\mathbb{Z}_p)$. We consider the case $N = 2$, $p = 2$ and $k = 2$.

2 Orientation

On any manifold M we can always put a riemannian metric. Let $\mathrm{O}(M) \to M$ be the orthonormal frame bundle with transition functions $\{\phi_{\alpha\beta}\}$ satisfying the 1-cocycle condition $\phi_{\alpha\beta}\phi_{\beta\gamma}\phi_{\gamma\alpha} = 1$. The question of orientability is whether we can reduce the structure group to $\mathrm{SO}(n)$ and construct the bundle of orientable orthonormal frames $\mathrm{SO}(M) \to M$. Let $\Delta_{\alpha\beta} = \det \phi_{\alpha\beta} = \pm 1$ then it follows from the cocycle condition that $\Delta_{\alpha\beta}\Delta_{\beta\gamma}\Delta_{\gamma\alpha} = 1$. The 1-cocycle $\{\Delta_{\alpha\beta}\}$ defines a line bundle. The cohomology class of this cocycle is denoted by $w_1 \in H^1(M,\mathbb{Z}_2)$ and is called the first Stiefel-Whitney class. The question is whether you can choose $\Delta_{\alpha\beta} = +1$ and in this way define an orientation, i.e., the orientation line bundle. Since $\det \phi_{\alpha\beta} = +1$, the structure group is reduced to $\mathrm{SO}(n)$. The way to do is to change the orientation of the framing for each open set

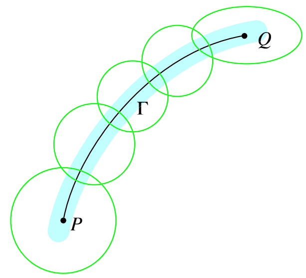

Figure 3: A tubular neighborhood of a path Γ from P to Q is covered by small contractible open sets.

V_α. If the transformation is given by $\psi_\alpha : V_\alpha \to \mathrm{O}(n)$ then the new cocycle is given by $\Delta'_{\alpha\beta} = (\det \psi_\alpha)^{-1} \Delta_{\alpha\beta} (\det \psi_\beta)$. Requiring $\Delta' = 1$ means that $\Delta_{\alpha\beta} = (\det \psi_\alpha)(\det \psi_\beta)^{-1}$, i.e., the 1-cocycle $\Delta_{\alpha\beta}$ is exact. The manifold M is orientable if the w_1 is trivial. Note that is the manifold is connected and simply connected then it is orientable.

Note that the space of riemannian metrics on M is an affine space because if g_0 and g_1 are metrics on M then $g_t = (1-t)g_0 + tg_1$ for $t \in [0,1]$ is a metric on M. Any homotopy of the metric cannot affect the transition functions since $\Delta_{\alpha\beta} = \pm 1$. The argument is independent of the choice of metric and therefore topological.

2.1 Orientation and Connections

We have a connected manifold with metric therefore we impose the torsion free riemannian (Levi-Civita) connection on the orthonormal frame bundle. The existence of a metric means that we can construct the bundle of orthonormal frames $\mathrm{O}(M) \to M$. Fix a point $P \in M$ and an orthonormal frame f_P at P. The choice of f_P determines an orientation at P. The question is whether at all other points in M we can can assign the same orientation of an orthonormal frame. Clearly in a small open set U we can assign a consistent orientation. The question is what happens as we string a whole bunch of open sets together. It is clear that it can be done in a tubular neighborhood of a fixed path Γ from P to Q as in Figure 3. We can parallel transport the frame f_P along the path Γ to Q. The orientation can be extended locally from the path Γ to the neighborhood depicted in Figure 3. Since M is connected, for each $Q \in M$ we can find a path Γ_Q from P to Q, by parallel transport we can assign an orientation to the frames at Q. Note that parallel transport gives an isometry $L(\Gamma_Q) : T_P M \to T_Q M$. The only issue is whether this assignment is well defined. Assume I chose a different path Γ'_Q.

Will I get the same orientation? The thing is to see if the isometries $L(\Gamma_Q)$ and $L(\Gamma'_Q)$ lead to the same orientation. The easiest way to think about this is to consider the isometry $L(\Gamma'_Q)^{-1}L(\Gamma_Q) = L({\Gamma'_Q}^{-1} \circ \Gamma_Q) : T_P \to T_P$ and compute $\det L({\Gamma'_Q}^{-1} \circ \Gamma_Q)$. If the answer is $+1$ then the two paths assigned the same orientation. Note that since ± 1 are disconnected two homotopic paths must both assign the same orientation. If the two paths are homotopic then the combined path ${\Gamma'_Q}^{-1} \circ \Gamma_Q$ is null homotopic. What we have constructed is a homomorphism from homotopy classes loops based at P to $\mathbb{Z}_2$. Let γ be a loop based at P then consider $\det L(\gamma)$. This actually defines a map from $\pi_1(M) \to \mathbb{Z}_2$. Note that since $\mathbb{Z}_2$ is abelian this induces a map $w_1 : H_1(M, \mathbb{Z}) \to \mathbb{Z}_2$ called the first Stiefel-Whitney class.

3 Spin Structures

We assume M is orientable for simplicity. All manifolds admit a riemannian metric and the induced bundle of orthonormal frames is $\mathrm{SO}(M) \to M$. The group $\mathrm{Spin}(n)$ is the universal cover of $\mathrm{SO}(n)$ and the question is whether we can lift the orthonormal frame bundle and construct the spin frame bundle $\mathrm{Spin}(M) \to M$. We can slightly generalize this problem. Let $\widetilde{G}$ be a connected, simply connected compact Lie group and let $\widetilde{Z} \subset \widetilde{G}$ be a subgroup of the center of $\widetilde{G}$. Let $G = \widetilde{G}/\widetilde{Z}$ then $\pi_1(G) = \widetilde{Z}$.

Let $P \to M$ be a principal G-bundle. Can we lift P to a principal $\widetilde{G}$-bundle $\widetilde{P} \to M$? In the case of spinors we have $G = \mathrm{SO}(n)$, $\widetilde{G} = \mathrm{Spin}(n)$ and $\widetilde{Z} = \mathbb{Z}_2$. Let $p : \widetilde{G} \to G$ be the covering transformation. If $g \in G$ then $p^{-1}(g) \subset \widetilde{G}$ contains $\#\widetilde{Z}$ elements. In plain language, there are $\#\widetilde{Z}$ possible lifts of g, and in particular $p^{-1}(1_G) = \widetilde{Z}$. Assume $\phi_{\alpha\beta} : V_\alpha \cap V_\beta \to G$ are the transition functions for $P \to M$. Remember that they satisfy $\phi_{\alpha\beta}\phi_{\beta\gamma}\phi_{\gamma\alpha} = 1_G$. We want to construct transition functions $\tilde{\phi}_{\alpha\beta}$ such that $\phi_{\alpha\beta} = p \circ \tilde{\phi}_{\alpha\beta}$ and $\tilde{\phi}_{\alpha\beta}\tilde{\phi}_{\beta\gamma}\tilde{\phi}_{\gamma\alpha} = 1_{\widetilde{G}}$.

Let $\tilde{\phi}_{\alpha\beta}$ be any lift of $\phi_{\alpha\beta}$. Note that any other lift will be of the form $\tilde{z}_{\alpha\beta}\tilde{\phi}_{\alpha\beta}$ where $\tilde{z}_{\alpha\beta} \in \widetilde{Z}$. In general

$$\tilde{\phi}_{\alpha\beta}\tilde{\phi}_{\beta\gamma}\tilde{\phi}_{\gamma\alpha} = \tilde{z}_{\alpha\beta\gamma} \qquad \text{where} \qquad \tilde{z}_{\alpha\beta\gamma} \in \widetilde{Z}.$$

The reason is that applying the projection p to the left hand side gives 1_G so the right hand side of the above must be in the center. Since the right hand side is in general not zero there is no principal bundle $\widetilde{P}$ that is a lift of P. You can convince yourself that on a quadruple overlap $V_\alpha \cap V_\beta \cap V_\gamma \cap V_\delta \neq$ the $\tilde{z}_{\alpha\beta\gamma}$ be a 2-cocycle:

$$\tilde{z}_{\alpha\beta\gamma}\tilde{z}_{\beta\gamma\delta}^{-1}\tilde{z}_{\gamma\delta\alpha}\tilde{z}_{\delta\alpha\beta}^{-1} = 1_{\widetilde{G}}\,.$$

Remember that if we change the lift $\tilde{\phi}_{\alpha\beta} \to \tilde{\phi}_{\alpha\beta}\tilde{z}_{\alpha\beta}$ and consequently $\tilde{z}_{\alpha\beta\gamma} \to \tilde{z}_{\alpha\beta\gamma}\tilde{z}_{\alpha\beta}\tilde{z}_{\beta\gamma}\tilde{z}_{\gamma\alpha}$. Let $[\tilde{z}_{\alpha\beta\gamma}] \in H^2(M, \widetilde{Z})$ be the cohomology class of the 2-cocycle $\{\tilde{z}_{\alpha\beta\gamma}\}$. The condition that $\widetilde{P} \to M$ exist is that the 2-cohomology class $[\tilde{z}_{\alpha\beta\gamma}]$ must vanish[2]. Assume the cohomology class vanishes then we can construct the transition functions $\{\phi_{\alpha\beta}\}$ for one such bundle. Can we constructs other lifts? The answer is yes. Assume $\{\tilde{z}_{\alpha\beta}\}$ is a 1-cocycle representing some cohomology class in $H^1(M, \widetilde{Z})$ then $\tilde{\phi}'_{\alpha\beta} = \tilde{z}_{\alpha\beta}\tilde{\phi}_{\alpha\beta}$ are good transition functions for some bundle $\widetilde{P}' \to M$. We conclude that the lifts of $P \to M$ are parametrized by $H^1(M, \widetilde{Z})$ and the obstruction to constructing the lift is in $H^2(M, \widetilde{Z})$.

The most famous example is the spin frame bundle. In this case the obstruction is denoted by $w_2 \in H^2(M, \mathbb{Z}_2)$ and is called the second Stiefel-Whitney class. The different spin frame bundles are classified by $H^1(M, \mathbb{Z}_2)$. For example, if the manifold is simply connected then $H^1(M, \mathbb{Z}_2)$ vanishes and the spin frame bundle is unique.

Note that this discussion does not address the question of the classification of $P \to M$ bundles. It just tells you how many $\widetilde{P} \to M$ bundles you can construct given a fixed $P \to M$ bundle.

3.1 Spin Structures and Connections

For simplicity we assume our manifold is connected and simply connected. This means that every loop is contractible. Since M is orientable, a metric gives us that bundle of oriented orthonormal frames $\mathrm{SO}(M) \to M$. Consider a map $\phi : S^2 \to M$ and let $N \in M$ be the image of the north pole, see Figure 4. We think of S^2 as the unit square and we will make an identification on the boundary such that ϕ maps the boundary of the square to the point N. If $(s,t) \in [0,1] \times [0,1]$ then we will think of s as the *selector* parameter which selects which loop, and t as the *time* parameter along the loop. The loop with $s = 0$ is the trivial loop, i.e., $\phi(0,t) = N$. As s increases the loops start growing, eventually pass by the south pole and then start to shrink until at $s = 1$ you have the trivial loop again. Let Γ_s be the loop described by $\phi(s,t)$. Parallel transport along Γ_s using the orthogonal connection on $\mathrm{SO}(M) \to M$ gives a map (holonomy) $H(\Gamma_s) : T_N M \to T_N M$ that we can think as a group element in $\mathrm{SO}(n)$. Note that $H(\Gamma_0) = H(\Gamma_1) = I$. We have constructed a map $\eta : S^1 \to \mathrm{SO}(n)$ given by $\eta(s) = H(\Gamma_s)$. Since $\pi_1(\mathrm{SO}(n)) = \mathbb{Z}_2$, you may find a homotopically non-trivial map. It is clear that the answer only depends on the homotopy class $[\eta]$ of the loop and on

[2] There is a lot of mathematics stuff going on with twisted K-theory and it is precisely what can you say if this cohomology class does not vanish.

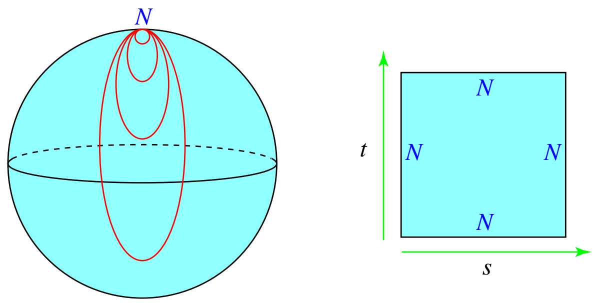

Figure 4: A family of loops on M. The family is generated by a map $\phi : S^2 \to M$. We think of S^2 as the unit square and we will make an identification on the boundary such that ϕ maps the boundary of the square to the point $N \in M$.The loops are labelled by a parameter s and the parameter t is time along the loop.

the homotopy class $[\phi]$ of the map $\phi : S^2 \to M$. Composition of parallel transport tells you that you have a group homomorphism from the abelian group $\pi_2(M)$ to the abelian group $\pi_1(\mathrm{SO}(n)) = \mathbb{Z}_2$. We have an induced map $w_2 : H_2(M) \to \mathbb{Z}_2$ called the second Stiefel-Whitney class. The issue here is whether you can define the bundle of spin frames $\mathrm{Spin}(M) \to M$. If w_2 is non-zero then you cannot get a well defined spinor frame because the answer depends on how you get there.

This construction becomes much more difficult if $\pi_1(M) \neq 0$ because then there will be different spin structures. Remember we already argued that the equivalence classes of spin structures are labelled by $H^1(M, \mathbb{Z}_2)$. This construction using the connection has to be adapted to single out which spin structure you are considering.

4 Classification of Bundles for $\dim M = 4$

The exact homotopy sequence tells you that

$$\cdots \to \pi_k(\mathbb{Z}_N) \to \pi_k(\mathrm{SU}(N)) \to \pi_k(SU(N)/\mathbb{Z}_N) \to \pi_{k-1}(\mathbb{Z}_N) \to \cdots$$

Applying this with $k = 1$ we have an isomorphism $\pi_1(\mathrm{SU}(N)/\mathbb{Z}_N) \approx \pi_0(\mathbb{Z}_N) = \mathbb{Z}_N$. Applying with $k = 3$ we have an isomorphism $\mathbb{Z} = \pi_3(\mathrm{SU}(N)) \approx \pi_3(\mathrm{SU}(N)/\mathbb{Z}_N)$. This last result tells us that the nontrivial topological 3-sphere is $\mathrm{SU}(N)/\mathbb{Z}_N$ is the same as the one in $\mathrm{SU}(N)$. Also we conclude that $\pi_0(\mathrm{SU}(N)/\mathbb{Z}_N) = \pi_2(\mathrm{SU}(N)/\mathbb{Z}_N) = \pi_4(\mathrm{SU}(N)/\mathbb{Z}_N) = 0$.

We begin with a good cover and a good simplicial decomposition of M. Let $E \to M$

be a $\mathrm{SU}(N)/\mathbb{Z}_N$ vector bundle. Choose any connection on this vector bundle and a basepoint P on M. For every point Q on the 1-skeleton of the simplicial decomposition choose a path to P. This can be done since M is connected. You can parallel transport the fiber of E over Q back to P using the connection and in this way you trivialize the bundle over the 1-skeleton.

Next we observe that since the bundle is trivial over each 1-simplex, we can identify the ends of the simplex and in this way we can think of the 1-simplex as a circle an we have a $\mathrm{SU}(N)/\mathbb{Z}_N$ bundle over S^1. Such a bundle is trivial and any two trivializations are related by a map $\xi^{(1)} : S^1 \to \mathrm{SU}(N)/\mathbb{Z}_N$. Therefore the trivializations are characterized by $\pi_1(\mathrm{SU}(N)/\mathbb{Z}_N) = \mathbb{Z}_N$. This means each trivialization I can assign a collection $\{e^{2\pi i n_{\alpha\beta}/N}\}$ for each 1-simplex $s^{(1)}_{\alpha\beta}$. Assume the 2-simplex $s^{(2)}_{\alpha\beta\gamma}$ has boundary $\partial s^{(2)}_{\alpha\beta\gamma} = s^{(1)}_{\alpha\beta} + s^{(1)}_{\beta\gamma} + s^{(1)}_{\gamma\alpha}$. The bundle is trivial on the boundary of the 2-simplex so it can be extended to the interior since a 2-simplex is contractible. This must be true since the bundle exists everywhere. Since we have identified all the fibers on the boundary of the 2-simplex, we effectively have a $\mathrm{SU}(N)/\mathbb{Z}_N$ bundle over S^2. Such bundles are characterized by giving the "equator patching map" which is a map $\xi^{(2)} : S^1 \to \mathrm{SU}(N)/\mathbb{Z}_N$. So to each 2-simplex I have to assign an element $e^{2\pi i n_{\alpha\beta\gamma}/N} \in \mathbb{Z}_N$. This is the $\mathbb{Z}_N$ monopole number. You can verify that if you change the trivialization on $s^{(1)}_{\alpha\beta}$ by $n_{\alpha\beta} \to (n_{\alpha\beta} + m_{\alpha\beta}) \mod N$ then $n_{\alpha\beta\gamma} \to (n_{\alpha\beta\gamma} + m_{\alpha\beta} + m_{\beta\gamma} + m_{\gamma\alpha}) \mod N$.

Next we look at the 3-skeleton. Let $s^{(3)}_{\alpha\beta\gamma\delta}$ be a three simplex with boundary $\partial s^{(3)}_{\alpha\beta\gamma\delta} = s^{(2)}_{\alpha\beta\gamma} - s^{(2)}_{\beta\gamma\delta} + \cdots$. Now $s^{(3)}_{\alpha\beta\gamma\delta}$ is topologically a 3-ball and $\partial s^{(3)}_{\alpha\beta\gamma\delta}$ is topologically S^2. We have constructed the bundle over the boundary S^2. The 3-ball is topologically contractible this means that any bundle over the ball must be trivial. This means that the bundle on the boundary must be trivial[3], i.e.,

$$n_{\alpha\beta\gamma} - n_{\beta\gamma\delta} + n_{\gamma\delta\alpha} - n_{\delta\alpha\beta} = 0 \mod N.$$

This means that $\{n_{\alpha\beta\gamma}\}$ defines a cocycle and the cohomology class $[n_{\alpha\beta\gamma}] \in H^2(M, \mathbb{Z}_N)$ of this cocycle characterizes the bundle. These are $\mathbb{Z}_N$ monopoles. We learn that the bundle has to be trivial on the boundary of each 3-simplex. This means that we can identify the boundary as a point and the 3-simplex becomes effectively an S^3. $\mathrm{SU}(N)/\mathbb{Z}_N$ bundles are trivial over S^3 since $\pi_2(\mathrm{SU}(N)/\mathbb{Z}_N) = 0$. The different trivializations on S^3 are given by a map $\xi^{(3)} : S^3 \to \mathrm{SU}(N)/\mathbb{Z}_N$ and these are characterized

[3]This construction should be contrasted with what happens in the case of a Dirac magnetic monopole. In that case we are in $\mathbb{R}^3$ with a singularity at the origin. On each S^2 of radius $r > 0$ we have a non-trivial bundle with Chern class given by $\int B$. Since the Chern class is integral it can't change as we shrink r so we find a singularity at $r = 0$. This should be contrasted with what happens if you have non-singular configurations. Note that the non-singular field strength of the standard instanton on S^4 can be thought to arise from a singular "nonabelian monopole" at the origin in $\mathbb{R}^5$.

by $\pi_3(\mathrm{SU}(N)/\mathbb{Z}_N) = \mathbb{Z}$. We can trivialize over the 3-skeleton. Finally we look at the 4-skeleton. A 4-simplex is topologically contractible this means that bundle on the boundary must be trivial and it is. We can identify all the fibers on the boundary and effectively we have a bundle over S^4. $\mathrm{SU}(N)/\mathbb{Z}_N$ bundles over S^4 are given by specifying the equator patching map $\xi^{(4)} : S^3 \to \mathrm{SU}(N)/\mathbb{Z}_N$. The topological classes of these maps are $\pi_3(\mathrm{SU}(N)/\mathbb{Z}_N) = \mathbb{Z}$. This means that we can assign an integer $n_{\alpha\beta\gamma\delta\epsilon}$ to each 4-simplex $s_{\alpha\beta\gamma\delta\epsilon}$. There is no 5-skeleton since we are on a 4-manifold so we can stop here. These integers give a homology class $[n_{\alpha\beta\gamma\delta\epsilon}] \in H^4(M, \mathbb{Z})$.

As far as I can tell to each $\mathrm{SU}(N)/\mathbb{Z}_N$ bundle E over a four manifold M we can assign two cohomology classes $\nu(E) \in H^2(M, \mathbb{Z}_N)$ and $\lambda(E) \in H^4(M, \mathbb{Z})$. I haven't thought about the converse.

5 Gauge Invariant $\mathbb{Z}_N$ Flux

Here I describe how to see the $\mathbb{Z}_N$ flux in a gauge invariant way. The basic scenario that has to be understood is the case of M being simply connected analogous to the discussion in Section 3.1. Assume we have a principal $P \to M$ bundle with structure group $\mathrm{SU}(N)/\mathbb{Z}_N$ and connection A. Consider a map $\phi : S^2 \to M$ which we represent as a square parametrized by $(s, t) \in [0, 1] \times [0, 1]$ as in Figure 4. I will think of s as the *selector* parameter which selects which loop, and t as the *time* parameter along the loop. If $x \in M$, then the fiber over x is denoted by P_x and it is isomorphic to $\mathrm{SU}(N)/\mathbb{Z}_N$. Let γ_x be a loop with base point x. Parallel transport along γ_x gives a map (holonomy) $H(\gamma_x) : P_x \to P_x$ that we can think as a group element in $\mathrm{SU}(N)/\mathbb{Z}_N$. Consider a family of paths Γ_s as described in Section 3.1. We note that $H(\Gamma_0) = H(\Gamma_1) = I$ where I is the identity in $\mathrm{SU}(N)/\mathbb{Z}_N$ (see Figure 5). Note that we have constructed a map $\eta : S^1 \to \mathrm{SU}(N)/\mathbb{Z}_N$ given by $\eta(s) = H(\Gamma_s)$. Note that η depends on the connection A but by its definition, η belongs to a homotopy equivalence class $[\eta] \in \pi_1(\mathrm{SU}(N)/\mathbb{Z}_N) \approx \mathbb{Z}_N$. Consequently, the connections will fall into homotopy equivalence classes labelled[4] by $\pi_1(\mathrm{SU}(N)/\mathbb{Z}_N)$. The *gauge invariant* $\mathbb{Z}_N$*-flux* is $[\eta]$ and it is determined by the connection and the homotopy class of the map $[\phi]$. Analogous to Section 3.1 we have a map $\nu : H_2(M, \mathbb{Z}) \to \pi_1(\mathrm{SU}(N)/\mathbb{Z}_N)$. In our case this equivalent to saying that $\nu \in H^2(M, \mathbb{Z}_N)$.

I would like to emphasize that there is a very important conceptual difference between $\mathbb{Z}_N$ flux in the non-abelian gauge theory and the abelian U(1) flux. Assume

[4] There may be other labels such as instanton number needed to fully label the homotopy equivalence classes of connections.

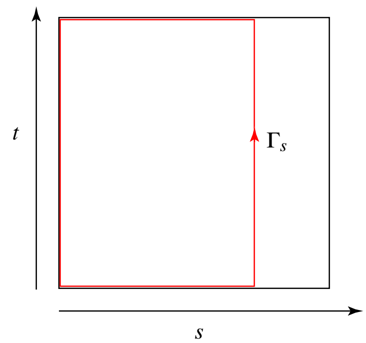

Figure 5: The loop Γ_s described by selector parameter s.

we have a U(1) connection then we can redo the argument in the previous paragraph. A difference is that we have a map $\eta : S^1 \to \mathrm{U}(1)$ with the property that $\eta(0) = \eta(1) = 1$. Note that $\eta(s) = \exp(i \int_{\Gamma_s} A)$. These maps are classified by $[\eta] \in \pi_1(\mathrm{U}(1)) = \mathbb{Z}$. There is a major difference that arises because we have an abelian connection. The observation is that

$$\lim_{s \to 1^-} \eta(s) = 1 \quad \text{tells us} \quad \lim_{s \to 1^-} \int_{\Gamma_s} A = 2\pi n \,.$$

The abelian nature of the connection actually tells us more $\int_{\phi(S^2)} F = 2\pi n$. The homotopy class of η determines the flux.

Things are very different in the $\mathrm{SU}(N)/\mathbb{Z}_N$ case. The notion of a non-abelian Stokes' Theorem is around but working with it is another question. We do have a notion of $\mathbb{Z}_N$ flux based on the holonomy. The integral type manipulations that we used in the abelian case do not exist but the analogy with $\eta(s)$ gives credence to a notion of non-abelian flux. To each $\phi : S^2 \to M$ we can assign a homotopy invariant in $\mathbb{Z}_N$.

This construction is much more difficult if $\pi_1(M) \neq 0$, i.e., *I don't really know how to do it.*

6 Detecting Torsion

I learned this from an old paper by D. Freed [2]. Let me make some background remarks first.

In this section I want to make some technical distinctions between three isomorphic cyclic groups of order N. The group $\mathbb{Z}_N \subset \mathrm{U}(1)$ is the abelian multiplicative cyclic group with elements $\{1, e^{2\pi i/N}, \ldots, e^{2\pi i(N-1)/N}\}$. The group $\mathbb{Z}/N\mathbb{Z}$ is the additive group $\{0, 1, \ldots, N-1\}$ with the group operation being addition modulo N. This follows from standard notation where $N\mathbb{Z} = \{0, \pm N, \pm 2N, \pm 3N, \ldots\}$ and we use the standard definition of the coset space $\mathbb{Z}/N\mathbb{Z}$. Let $\mathbb{Q} \subset \mathbb{R}$ be the additive group of the rational numbers. I want to describe the group $\mathbb{Z}/N \subset \mathbb{Q}/\mathbb{Z}$. The elements of $\mathbb{Z}/N$ are taken to be

$$\left\{0, \frac{1}{N}, \frac{2}{N}, \ldots, \frac{N-1}{N}\right\}$$

where the group operation is addition modulo 1. Note that $\mathbb{Q}/\mathbb{Z}$ is the additive group of rational numbers in $[0, 1)$ with the group operation being addition modulo 1.

A motivation for the above discussion is to consider the case of $\mathbb{Z}/n\mathbb{Z} \oplus \mathbb{Z}/m\mathbb{Z}$ versus $\mathbb{Z}/n \oplus \mathbb{Z}/m$. In the former when considering the group operation, we have to keep track that things are $\mod n$ in the first factor and $\mod m$ in the second factor. In the latter case everything is $\mod 1$ and the bookkeeping is easier.

Assume G is a group and $K \subset G$ is a normal subgroup, then it is a fundamental theorem of group theory that there exists a group H and a group homomorphism $\phi : G \to H$ such that $K = \ker G$. A consequence of this is that there is a short exact sequence $0 \to K \to G \xrightarrow{\phi} H \to 0$. In fact, it is easy to see that $H \approx G/K$. Note that if G is abelian then any subgroup K is normal and therefore you always have a short exact sequence $0 \to K \to G \to G/K \to 0$.

Let G be an abelian group. An element $g \in G$ is said to be torsion if there exists $N \in \mathbb{Z}$ such that $Ng = 0$. Note that $\operatorname{Tor} G$, the set of all torsion elements, is a subgroup. The proof follows from the observation that if $N_1 g_1 = 0$ and $N_2 g_2 = 0$ then $N_1 N_2 (g_1 + g_2) = 0$. Since G and $\operatorname{Tor} G$ are abelian we have that $\operatorname{Free} G = G/(\operatorname{Tor} G)$ is a group and we have the short exact sequence

$$0 \to \operatorname{Tor} G \to G \to \operatorname{Free} G \to 0\,. \tag{6.1}$$

Note that given $g \in G$, the free part of g, denoted by g^{free} is well defined and given by the projection[5] $g^{\text{free}} = g + \operatorname{Tor} G$. For generic $g \in G$ there is no canonical way of specifying the component of g in $\operatorname{Tor} G$, see Figure 6.

It follows from the above that we automatically have a short exact sequence

$$0 \to \operatorname{Tor} H^k(M, \mathbb{Z}) \to H^k(M, \mathbb{Z}) \to \operatorname{Free} H^k(M, \mathbb{Z}) \to 0\,. \tag{6.2}$$

Let $[\omega] \in H^k(M, \mathbb{Z})$ and let $[\omega]^{\text{free}} \in \operatorname{Free} H^k(M, \mathbb{Z})$ be its well defined free part. The DeRham theorem tells you that there is a closed k-form ω that represents $[\omega]^{\text{free}}$.

[5]We in an abelian group so a coset is written additively.

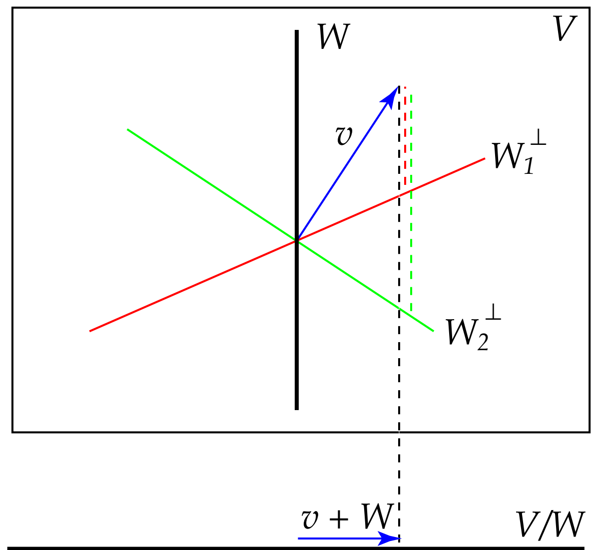

Figure 6: Let V be a vector space and W a vector subspace. Remember that V/W is the vector space defined by the equivalence relation $v_1 \sim v_2$ if $v_1 - v_2 \in W$. The coset associated with the equivalence class of v is written $v + W$. There is a canonical projection $\pi : V \to V/W$. given by $\pi : v \mapsto v + W$ and a short exact sequence $0 \to W \to V \xrightarrow{\pi} V/W \to 0$, i.e., $\ker \pi = W$. The lesson here is that given $v \in V$ and $W \subset V$ there is no canonical way to project v to W. There is a canonical notion of projection *along* W given by π. Decompose v we need a complementary subspace $W^\perp$ such that $V = W \oplus W^\perp$. There are many choices of $W^\perp$ as illustrated in the figure by $W_1^\perp$ and $W_2^\perp$. The projection of v *into* W requires the complementary subspace and there is no canonical construction unless you have some extra structure such as a metric on V that gives the orthogonal projection. In summary, given $W \subset V$ there is no canonical way to state what is the component of v along W unless $v \in W$.

If z is a integral k-cycle then the cohomology-homology pairing is given by

$$\left\langle [\omega]^{\text{free}}, [z] \right\rangle = \left(\int_z \omega \right) \in \mathbb{Z}\,.$$

DeRham cohomology cannot detect torsion. The reason is that DeRham cohomology is real cohomology $H^k(M,\mathbb{R})$ since the integral of a generic closed k-form over an integral cycle does not have to be an integer. Algebra tells you that $H^k(M,\mathbb{R}) = \mathbb{R} \otimes H^k(M,\mathbb{Z})$, i.e., replace "integer scalars" by "real scalars". Assume $[\tau] \in H^k(M,\mathbb{Z})$ is a torsion element, $N[\tau] = 0$, then with "real scalars" we can write $[\tau] = 1[\tau] = \frac{1}{N} \cdot N[\tau] = 0$. In summary, tensoring by $\mathbb{R}$ kills the torsion. Said differently we have that $N\tau = d\phi$ and $\int_z \tau = \frac{1}{N}\int_z d\phi = \frac{1}{N}\int_{\partial z} \phi = 0$.

6.1 Miscellaneous Homology/Cohomology Stuff

I will fix the manifold and choose a good simplicial decomposition. In homology we have the following: the k-chains with integer coefficients will be denoted by C_k, the k-cycles will be denoted by $Z_k = \ker \partial_k$, and the k-dimensional boundaries will be denoted by $B_k = \operatorname{im} \partial_{k+1} = \partial C_{k+1}$. The k-th homology group is given by $H_k = Z_k/B_k = (\ker \partial_k)/(\operatorname{im} \partial_{k+1})$. This is the object we have normally written as $H_k(M,\mathbb{Z})$.

Let $(e_1, e_2, \ldots, e_r)$ be a basis for C_k then a general k-chain is of the form $k_1 e_1 + \cdots + k_r e_r$ where $k_j \in \mathbb{Z}$. This tells you that C_k is isomorphic[6] to the free abelian group $\mathbb{Z}^r$. When you look at $H_k = Z_k/B_k$ you no longer get a free group but something of the form $\mathbb{Z}^m \oplus \mathbb{Z}/n_1 \oplus \cdots \oplus \mathbb{Z}/n_l$ as we saw in the case of the Klein bottle in Section 1.2. A nice thing about free abelian groups is that "linear algebra" is very much like linear algebra for vector spaces. First we define the cochains as the integer valued "linear functionals", i.e., the "dual space": $C^k = \operatorname{Hom}(C_k, \mathbb{Z})$. I think that it is clear that $\dim C^k = \dim C_k$ and that C^k is a free abelian group. Given the sequence of maps $\partial_k : C_k \to C_{k-1}$ there are the adjoint maps $\delta_k : C^k \to C^{k+1}$ defined by $(\delta_k \omega)(c) = \omega(\partial_{k+1} c)$. This leads to the sequences

$$\begin{array}{ccccccccc}
\cdots & \longleftarrow & C_{k-1} & \xleftarrow{\partial_k} & C_k & \xleftarrow{\partial_{k+1}} & C_{k+1} & \longleftarrow & \cdots \\
\cdots & \longrightarrow & C^{k-1} & \xrightarrow{\delta_{k-1}} & C^k & \xrightarrow{\delta_k} & C^{k+1} & \longrightarrow & \cdots
\end{array}$$

The k-cocycles are defined to be $Z^k = \ker \delta_k$ and the k-coboundaries are defined to be $B^k = \operatorname{im} \delta_{k-1} = \delta_{k-1} C^{k-1}$. The k-th cohomology group is $H^k = Z^k/B^k = (\ker \delta_k)/(\operatorname{im} \delta_{k-1})$. H^k and H_k are in general not dual spaces because in general H_k

[6] In a different triangulation the dimensionality of C_k will generally change. The big theorem is that the homology is independent of the simplicial decomposition.

and H^k are not free abelian groups. Note that the only group homomorphism from $\mathbb{Z}/n$ to $\mathbb{Z}$ is the zero homomorphism. This means that $\operatorname{Tor} H_k$ "plays no role" in $\operatorname{Hom}(H_k, \mathbb{Z})$, i.e., $\operatorname{Hom}(H_k, \mathbb{Z})$ is entirely determined by $\operatorname{Hom}(\operatorname{Free} H_k, \mathbb{Z})$. In fact, the universal coefficients theorem says that the free parts are dual spaces of each other: $\operatorname{Free} H^k \approx \operatorname{Hom}(\operatorname{Free} H_k, \mathbb{Z})$. The situation is more delicate for the torsion parts where the universal coefficients theorem states that there is a shift by one: $\operatorname{Tor} H^k = \operatorname{Hom}(\operatorname{Tor} H_{k-1}, \mathbb{Q}/\mathbb{Z})$. Notice that we are now studying linear functionals with values in $\mathbb{Q}/\mathbb{Z}$.

What is an element of $\operatorname{Hom}(\operatorname{Tor} H_{k-1}, \mathbb{Q}/\mathbb{Z})$? Choose $\varpi \in \operatorname{Hom}(C_{k-1}, \mathbb{Q})$, namely, a linear functional with "rational coefficients", then the induced homomorphism $\varphi \in \operatorname{Hom}(\operatorname{Tor} H_{k-1}, \mathbb{Q}/\mathbb{Z})$ is given by $\varphi([z]) = \varpi(z) \mod 1$ for every $[z] \in \operatorname{Tor} H_{k-1}$ with the proviso that things have to be independent of representative of $[z]$ chosen. This means that

$$\varpi(z + \partial x) = \varpi(z) \mod 1\,,$$

for every $[z] \in \operatorname{Tor} H_{k-1}$ and $x \in C_k$. This tells you that $\varpi(\partial x) = (d\varpi)(x) = 0 \mod 1$. We are now at the tricky part. The statement that $(d\varpi)(x) = 0 \mod 1$ for all $x \in C_k$ tells you that $\eta = d\varpi$ is an integer cochain! In fact since $d^2 = 0$ we have that $\eta \in Z^k$. Note that in general there does not exist an *integer* cochain $\nu \in C^{k-1}$ such that $\eta = d\nu$ though there is a rational cochain $\varpi \in \operatorname{Hom}(C_{k-1}, \mathbb{Q})$ such $\eta = d\varpi$. In fact there exists an integer L such that $L\varpi$ is an integer cochain which implies that $L\eta = d(L\varpi)$ and consequently the cohomology class of the integer cochain $L\eta$ is trivial, i.e., $L[\eta] = 0$, and we conclude that $[\eta] \in \operatorname{Tor} H^k$.

Next we use the fact that $[z]$ is a torsion element which means that there exists an integer N and a chain $y \in C_k$ such that $Nz = \partial y$, i.e., $N[z] = 0$. One more remark is that $\partial\bigl(\frac{1}{N}\, y\bigr) = z \in C_{k-1}$ and $\frac{1}{N}\, y \in C_k(\mathbb{Q})$. Since z is integral we have that $\partial\bigl(\frac{1}{N}\, y\bigr) = 0 \mod 1$ and therefore $\frac{1}{N}\, y \in Z_k(\mathbb{Q}/\mathbb{Z})$ and represents a homology class $\bigl[\frac{1}{N}\, y\bigr] \in H_k(\mathbb{Q}/\mathbb{Z})$. Summarizing, given $[z] \in \operatorname{Tor} H_{k-1}$ you can construct $\bigl[\frac{1}{N}\, y\bigr] \in H_k(\mathbb{Q}/\mathbb{Z})$.

Note that $(N\varpi)(z) = \varpi(Nz) = \varpi(\partial y) = (d\varpi)(y) = \eta(y) \in \mathbb{Z}$. From this we conclude that $\varpi(z) = \frac{1}{N}\, \eta(y) \in \mathbb{Q}$, and consequently

$$\varphi([z]) = \varpi(z) \mod 1 = \frac{1}{N}\, \eta(y) \mod 1 = \eta\left(\frac{1}{N}\, y\right) \mod 1 \in \mathbb{Q}/\mathbb{Z}\,. \tag{6.3}$$

There are three things we still have to verify to make sure the equation above represents the isomorphism $\operatorname{Tor} H^k \approx \operatorname{Hom}(\operatorname{Tor} H_{k-1}, \mathbb{Q}/\mathbb{Z})$. First, we have to verify that everything is independent of the representative chosen for $[z]$. Let $z' = z + \partial v$ for $v \in C_k$ then $\varpi(z') = \varpi(z) + \varpi(\partial v) = \frac{1}{N}\, \eta(y) + (d\varpi)(v) = \frac{1}{N}\, \eta(y) + \eta(v)$. Now $\eta(v) \in \mathbb{Z}$

so when we do the modulo 1 reduction to $\mathbb{Q}/\mathbb{Z}$ the $\eta(v)$ term is lost so we conclude that $\varpi(z') = \varpi(z) \mod 1$. Second, we have to verify that things are independent of the choice of y. If y and y' are such that $Nz = \partial y = \partial y'$ then $\partial(y' - y) = 0$ and we conclude that $y' = y + \tilde{z}$ where $\tilde{z} \in C_k$. In terms of rational coefficients we have that $\eta(\tilde{z}) = (d\varpi)(\tilde{z}) = \varpi(\partial \tilde{z}) = 0$ and so we conclude that $\eta(y') = \eta(y)$. Finally we have to verify that things are independent of the choice of η. So far we have established that $\eta \in Z^k$ and $[\eta] \in \operatorname{Tor} H^k$ but we have not established that (6.3) only depends on the cohomology class of η. To verify this consider $\eta' = \eta + d\lambda$ where $\lambda \in C^{k-1}$. Note that $\frac{1}{N}(d\lambda)(y) = \lambda\bigl(\frac{1}{N}\partial y\bigr) = \lambda(z) \in \mathbb{Z}$ since λ and z have integer coefficients. We see that $\frac{1}{N}\eta'(y) = \frac{1}{N}\eta(y) \mod 1$. Clearly all we have done is reversible. In summary, we have established the isomorphism $\operatorname{Tor} H^k \approx \operatorname{Hom}(\operatorname{Tor} H_{k-1}, \mathbb{Q}/\mathbb{Z})$ by finding the explicit map.

How do we detect torsion? The idea is to use the isomorphism we established $\operatorname{Tor} H^k \approx \operatorname{Hom}(\operatorname{Tor} H_{k-1}, \mathbb{Q}/\mathbb{Z})$. Namely, to every $[\eta] \in \operatorname{Tor} H^k$ these corresponds a $\varphi \in \operatorname{Hom}(\operatorname{Tor} H_{k-1}, \mathbb{Q}/\mathbb{Z})$ and all we we have to do is exhibit φ.

Example 1. This example is from [2] but I find his discussion is too telegraphic and incomprehensible. Assume we have a line bundle $L \to M$ with connection A and curvature F. We wish to detect $\operatorname{Tor} H^2$. Let $[z] \in \operatorname{Tor} H_1$ then from our previous discussion we know that there exists $y \in C_2$ such that $Nz = \partial y$. Parallel transport about z gives holonomy $e^{-i\int_z A}$ from this we can construct a homomorphism $\varpi' \in \operatorname{Hom}(C_1, \mathbb{R}/\mathbb{Z})$ by defining

$$\varpi'(z) = -\frac{1}{2\pi}\int_z A \mod 1\,.$$

The problem with this homomorphism is that in general it does not give a rational number so we will have to modify it. To modify it appropriately we make a naive incorrect computation that states that $\int_{Nz} A = \int_{\partial y} A = \int_y dA = \int_y F$. In fact, in some sense the point my paper [3] is that such a naive computation is incorrect because A is defined only locally. You can verify this by observing that if we choose a different y' such that $Nz = \partial y'$ then as previously discussed we have that $y' = y + \tilde{z}$ where $\tilde{z} \in Z_2$. Redoing the above computation we would conclude that $\int_{Nz} A = \int_{y'} F$ but $\int_{y'-y} F = \int_z F = 2\pi n_{[z]}$ where $n_{[z]} \in \mathbb{Z}$. In the naive manipulations there is an ambiguity in $2\pi\mathbb{Z}$. This means that

$$\frac{1}{2\pi}\int_{Nz} A = \frac{1}{2\pi}\int_y F \mod 1\,.$$

Motivated by the above discussion we consider the homomorphism

$$\varpi(z) = \left(-\frac{1}{2\pi}\int_z A + \frac{1}{N}\frac{1}{2\pi}\int_y F\right) \mod 1 \in \mathbb{Z}/N \subset \mathbb{Z}/\mathbb{Q}\,.$$

A brief computation shows $\varpi(z)$ on depends to $[z] \in \operatorname{Tor} H_1$ so we have constructed $\varphi \in \operatorname{Hom}(\operatorname{Tor} H_1, \mathbb{Q}/\mathbb{Z})$ by defining

$$\varphi([z]) = \varpi(z) = \left(-\frac{1}{2\pi} \int_z A + \frac{1}{N} \frac{1}{2\pi} \int_y F \right) \mod 1 . \tag{6.4}$$

By the general theorem φ represents a torsion class $[\eta] \in \operatorname{Tor} H^2$. If we want to be more accurate we note that $\varphi \in \operatorname{Hom}(\operatorname{Tor} H_1, \mathbb{Z}/N) \subset \operatorname{Hom}(\operatorname{Tor} H_1, \mathbb{Q}/\mathbb{Z})$.

7 Classifying Spaces

7.1 Classifying spaces for vector bundles

This is my recollection of explanations by I. M. Singer in the past. Let $\mathbb{F}$ be a field, either $\mathbb{R}$ or $\mathbb{C}$. The Grassmann manifold $\operatorname{Gr}_k(\mathbb{F}^N)$ is the manifold of k-planes in $\mathbb{F}^N$, e.g., $\operatorname{Gr}_k(\mathbb{R}^N) \approx \mathrm{SO}(N)/(\mathrm{SO}(k) \times \mathrm{SO}(N-k))$. There is a tautological vector bundle Q of rank k over $\operatorname{Gr}_k(\mathbb{F}^N)$. Namely, a point $p \in \mathbb{F}^N$ is a k-dimensional plane. The fiber Q_p at this point is precisely that k-dimensional plane.

Let M be a manifold and let E be a rank k $\mathbb{F}$-vector bundle over M. There is a theorem that states that there exists a vector bundle $E^\perp \to M$ with the property that $E \oplus E^\perp \approx M \times \mathbb{F}^N$ for some sufficiently large N. Namely, you can always find a vector bundle $E^\perp$ of sufficiently large rank such that $E \oplus E^\perp$ is a trivial bundle. The theorem is stronger. It actually states that for all vector bundles of rank k over M you can determine a large enough N that works universally. I will now describe a map $\varphi : M \to \operatorname{Gr}_k(\mathbb{F}^N)$. At a point $x \in M$ we have the fiber E_x. Since $E \oplus E^\perp = M \times \mathbb{F}^N$ we have that E_x is a k-plane in $\mathbb{F}^N$. The image of x under φ is precisely that k-plane. By construction we have that the pull-back of the tautological bundle $\varphi^* Q$ is precisely E. The big theorem is that all rank k vector bundles arise from a map φ and that homotopic maps lead to isomorphic vector bundles. This is the sense that homotopy equivalence classes of maps $[M, \operatorname{Gr}_k(\mathbb{F}^N)]$ classify all rank k $\mathbb{F}$-vector bundles over M.

It is possible to make estimates for how large an N you need but many people find it convenient to take an inductive limit and consider $N = \infty$ so you get the BG type of classifying spaces that correspond to $\operatorname{Gr}_k(\mathcal{H}_{\mathbb{F}})$ for some Hilbert space $\mathcal{H}_{\mathbb{F}}$.

How do you prove the $E \oplus E^\perp$ theorem? Assume M is a compact manifold and let $\{V_\alpha\}$ be a good cover for M, and let $\{\rho_\alpha\}$ be a partition of unity subordinate to the cover. This means that $\rho_\alpha : V_\alpha \to [0,1]$, ρ_α has compact support in V_α, and $\sum_\alpha \rho_\alpha = 1$. Since V_α is contractible, the bundle restricted to V_α is trivial and we can find k non-vanishing sections $\{(s_1)_\alpha, (s_2)_\alpha, \ldots, (s_k)_\alpha\}$ that span the vector bundle over V_α. We don't care what these sections do outside of V_α. Consider the set $\cup_\alpha\{\rho_\alpha(s_1)_\alpha, \rho_\alpha(s_2)_\alpha, \ldots, \rho_\alpha(s_k)_\alpha\} = \{s_i\}_{i=1}^N\}$. Note that $N < \infty$ because M is compact if not we could consider $N = \infty$. This finite number of global sections of E span the fiber at each point. Remember that the space of sections of E is an infinite dimensional vector space. This subset of sections $\{s_i\}_{i=1}^N$ spans an N-dimensional vector subspace W of the space of sections. Note that there is a map $\mathrm{ev}_x : W \to E_x$ given by

$$\mathrm{ev}_x : a_1 s_1 + \cdots + a_N s_N \mapsto a_1 s_1(x) + \cdots + a_N s_N(x) \in E_x \,.$$

We define $\ker \mathrm{ev}_x = E_x^\perp$ then we immediately have that $E_x \oplus E_x^\perp = \mathbb{F}^N$. Doing this over all points in M we see that $E \oplus E^\perp = M \times \mathbb{F}^N$ because it is the same $\mathbb{F}^N$ everywhere.

7.2 Classifying spaces for principal bundles

This needs a little bit more work.

By doing a variant on the construction in the previous section you can do a classification for principal bundles. The idea is that if you have a metric on $\mathbb{F}^N$ then this induces a metric on each E_x. A metric on the vector space E_x allows you to choose an orthonormal frame. Any other orthonormal frame is obtained by the appropriate unitary/orthogonal group. Let $P \to M$ be a principal fiber bundle with structure group G which is either $\mathrm{U}(k)$ or $\mathrm{SO}(k)$ with associated vector bundle $E \to M$. The Stiefel manifold $\mathrm{V}_k(\mathbb{F}^N) \approx \mathrm{SO}(N)/\,\mathrm{SO}(N-k)$ is the set of all orthonormal k-frames in $\mathbb{F}^N$. This manifold has a tautological principal bundle $\mathcal{F} \to \mathrm{V}_k(\mathbb{F}^N)$ with structure group G. For a frame $f \in \mathrm{V}_k(\mathbb{F}^N)$, the fiber $\mathcal{F}_x$ is the set of all orthonormal k-frames for the k-plane determined by f. All these different frames are related by a G transformation. This principal bundle is denoted by EG and the base by BG. The claim is that all principal G-bundles over M are classified by homotopy equivalence classes of maps $[M, BG]$.

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